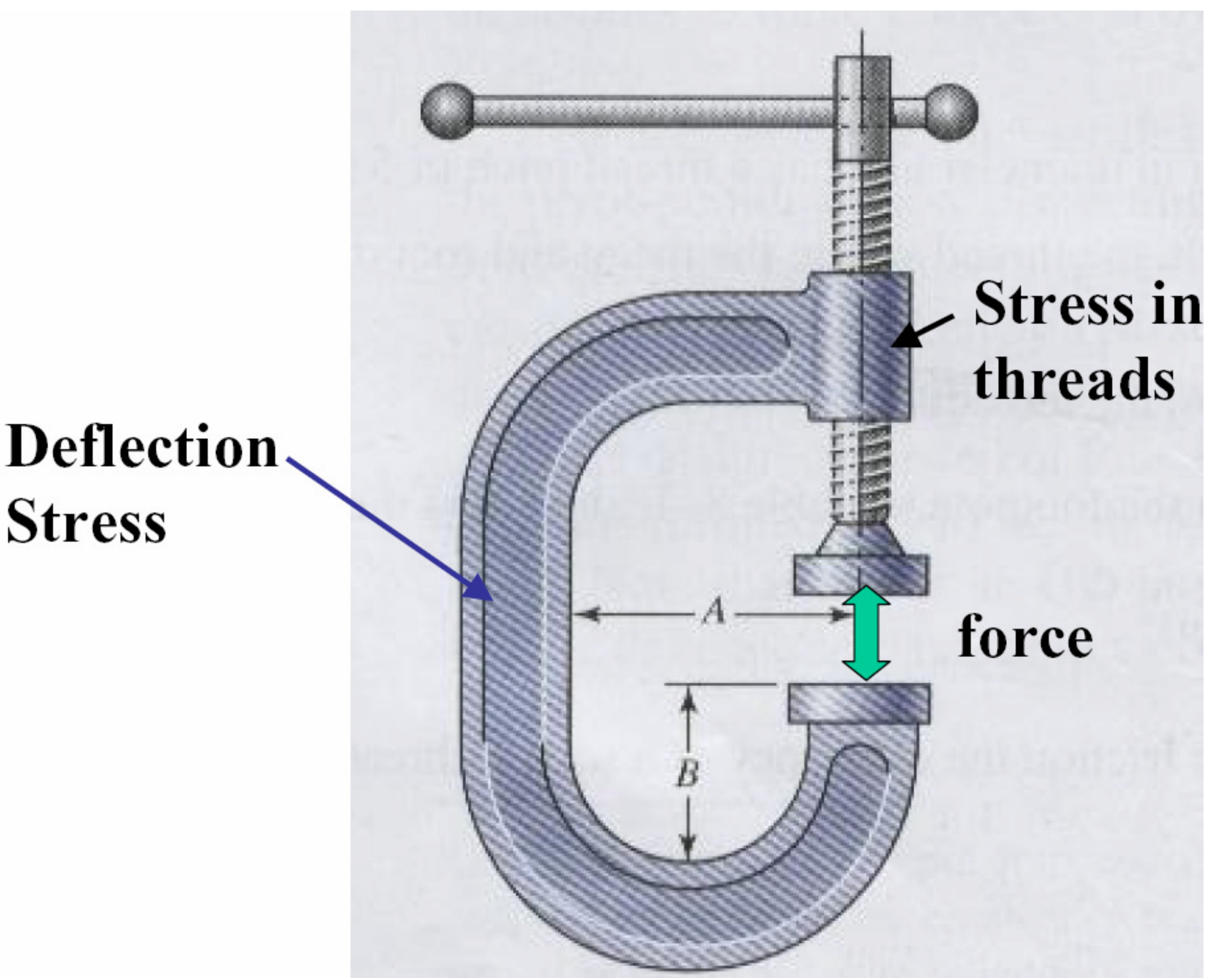
Design considerations: Stress and deflection analysis

For any application, components need to sustain a loading => stress and deflection

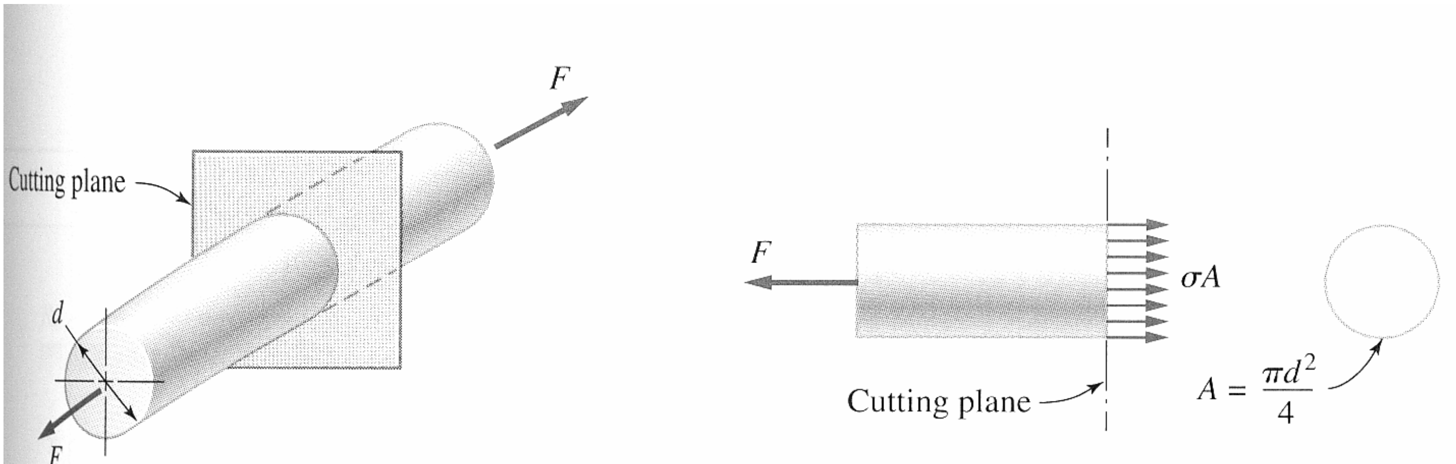


Loading

- **Types of loading**
  - Tensile
  - Compressive
  - Flexural (bending)
  - Shear
  - Torsional
  - Combined
- **Loading modes**
  - Static (preloaded bolt, pressure vessel, deadweight)
  - Cyclic (gear teeth, shafts, springs)
  - Impact (contact time < ½ natural frequency^{-1})

Stresses

- **Axial stress** (tensile + or compressive -)



(*a*) Tensile force on solid cylindrical member.

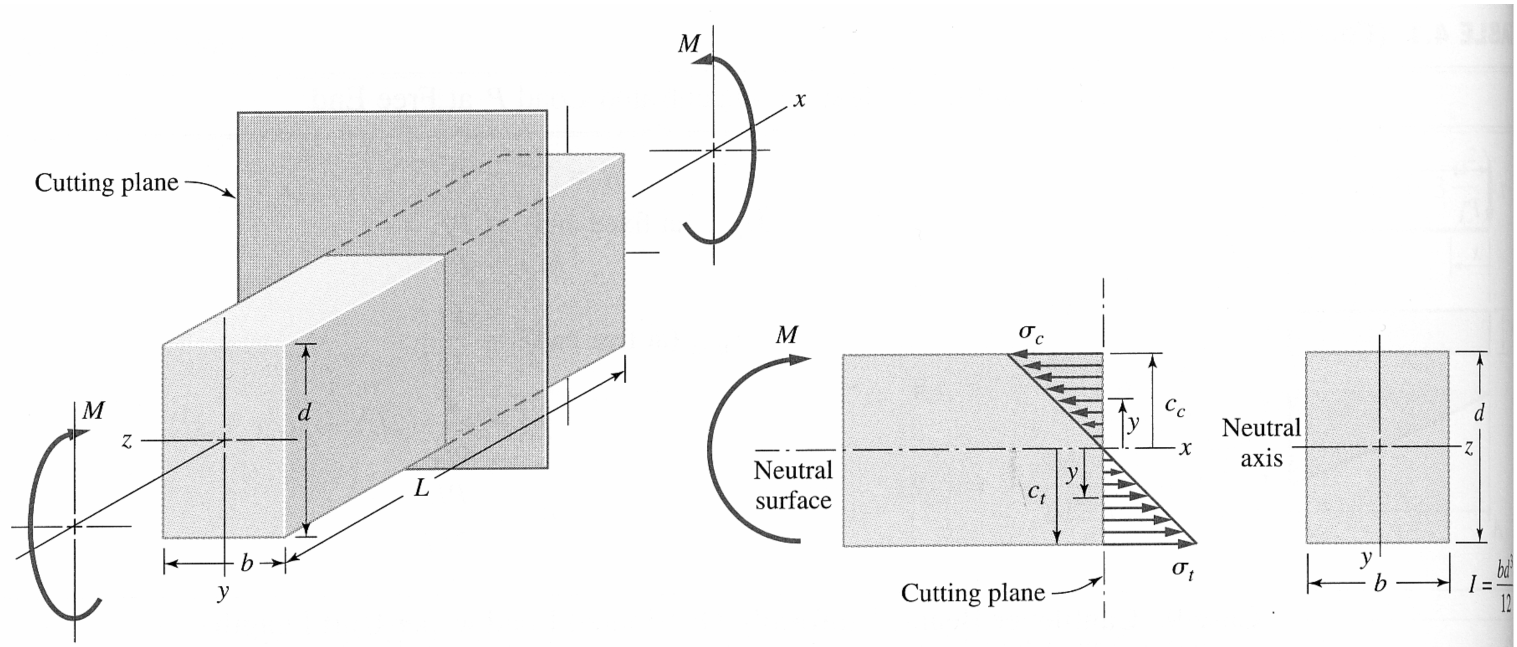
(*b*) Side view of front free body showing uniform normal stress distribution over cut surface of cylinder, producing the force σA.

Collins, Fig. 4.2

$$\sigma = \frac{F}{A}$$

Stresses

- **Bending stress in straight beams**



(*a*) Pure bending moment on a beam of rectangular cross section.
(*b*) Side view of front free body showing linear normal stress distribution across cut surface.

Collins, Fig. 4.3

$$\sigma = \frac{My}{I}$$

$$\sigma_t = \frac{Mc_t}{I} \qquad \sigma_c = -\frac{Mc_c}{I} \qquad I = \frac{bd^3}{12}$$

M : applied bending moment

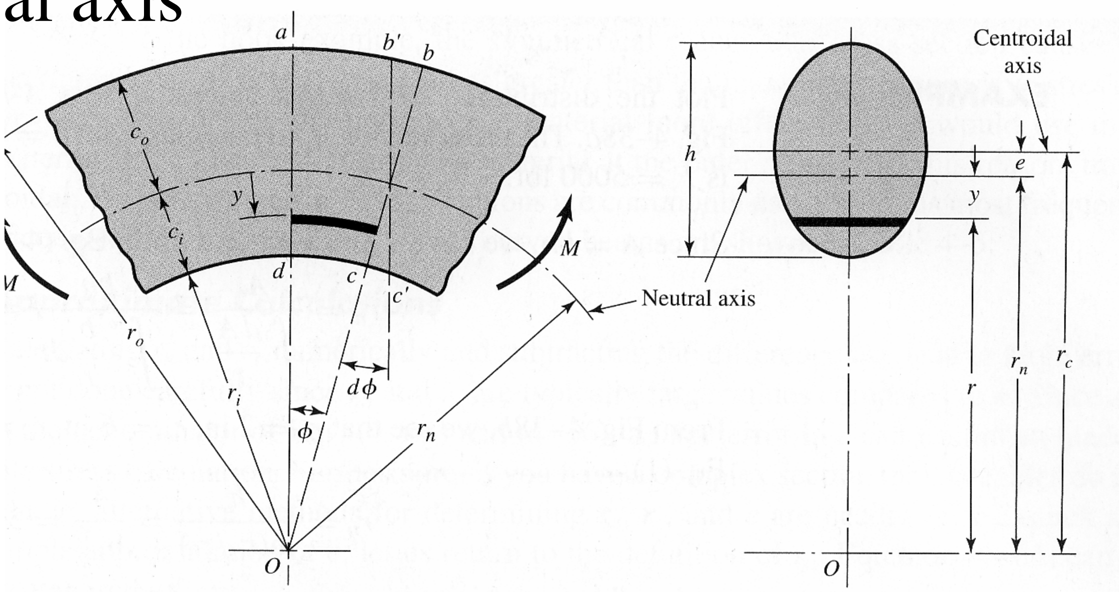
I : moment of inertia of cross-section about neutral axis

y, c_t, c_c : distances from neutral axis

Stresses

- **Bending stress in initially curved beams**

Moment calculated about centroidal axis, not neutral axis



h = depth of section

c_o = distance from neutral axis to outer fiber

c_i = distance from neutral axis to inner fiber

r_n = radius of neutral axis

r_c = radius of centroidal axis

e = distance from centroidal axis to neutral axis

M = bending moment; positive M decreases curvature

Shigley, Fig. 3-34

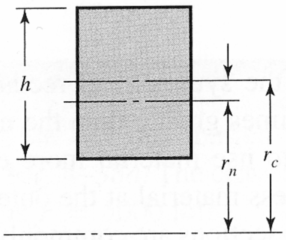
$$e = r_c - r_n$$

Inside stress: Ouside stress:

$$\sigma_i = \frac{Mc_i}{Aer_i} \qquad \sigma_o = -\frac{Mc_o}{Aer_o} \quad \text{(pure bending case)}$$

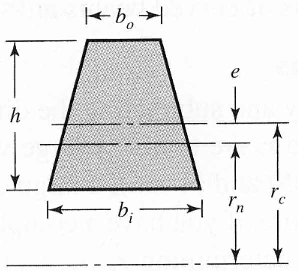
Shigley Table 3-4

Formulas for Sections of Curved Beams



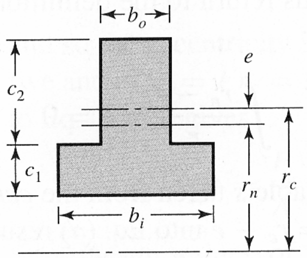
$$r_c = r_i + \frac{h}{2}$$

$$r_n = \frac{h}{\ln(r_o/r_i)}$$



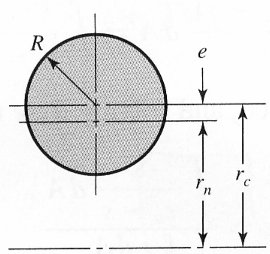
$$r_c = r_i + \frac{h}{3}\,\frac{b_i + 2b_o}{b_i + b_o}$$

$$r_n = \frac{A}{b_o - b_i + [(b_i r_o - b_o r_i)/h]\ln(r_o/r_i)}$$



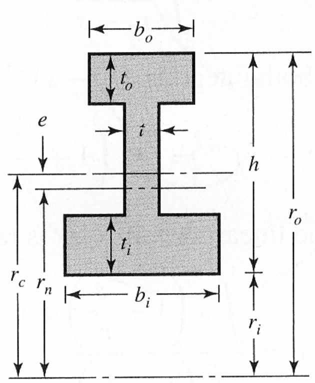
$$r_c = r_i + \frac{b_i c_1^2 + 2b_o c_1 c_2 + b_o c_2^2}{2(b_o c_2 + b_i c_1)}$$

$$r_n = \frac{b_i c_1 + b_o c_2}{b_i \ln[(r_i + c_1)/r_i)] + b_o \ln[r_o/(r_i + c_1)]}$$



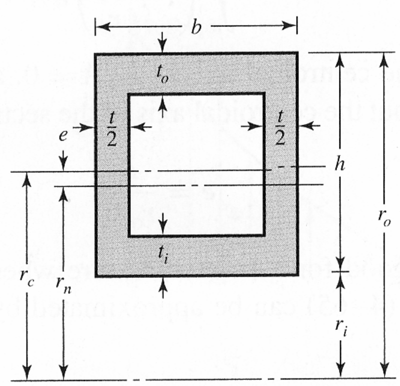
$$r_c = r_i + R$$

$$r_n = \frac{R^2}{2\left(r_c - \sqrt{r_c^2 - R^2}\right)}$$



$$r_c = r_i + \frac{\frac{1}{2}h^2 t + \frac{1}{2}t_i^2(b_i - t) + t_o(b_o - t)(h - t_o/2)}{t_i(b_i - t) + t_o(b_o - t) + ht}$$

$$r_n = \frac{t_i(b_i - t) + t_o(b_o - t) + ht_o}{b_i \ln\dfrac{r_i + t}{r_i} + t\ln\dfrac{r_o - t_o}{r_i + t_i} + b_o \ln\dfrac{r_o}{r_o - t_o}}$$

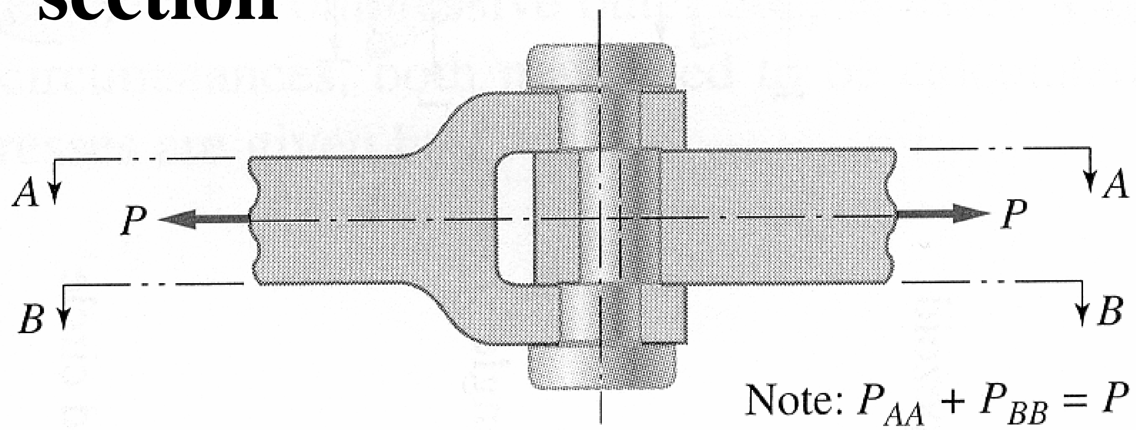


$$r_c = r_i + \frac{\frac{1}{2}h^2 t + \frac{1}{2}t_i^2(b - t) + t_o(b - t)(h - t_o/2)}{ht + (b - t)(t_i + t_o)}$$

$$r_n = \frac{(b - t)(t_i + t_o) + ht}{b\left(\ln\dfrac{r_i + t_i}{r_i} + t\ln\dfrac{r_o}{r_o + t_o}\right) + t\ln\dfrac{r_o - t_o}{r_i + t_i}}$$

Stresses

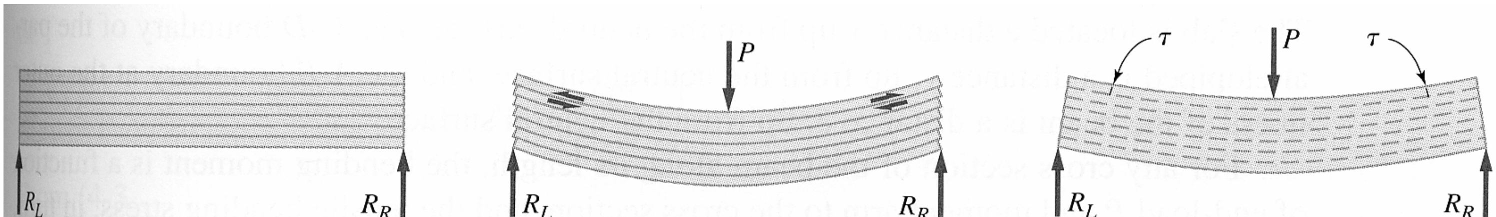
- **Direct shear: stress parallel to section**



Collins, Fig. 4.4

$$\tau = \frac{P}{A} = \frac{P}{\pi r^2}$$

- **Transverse shear stress**



(*a*) Unloaded beam made of a stack of individual laminae free to slide over each other.

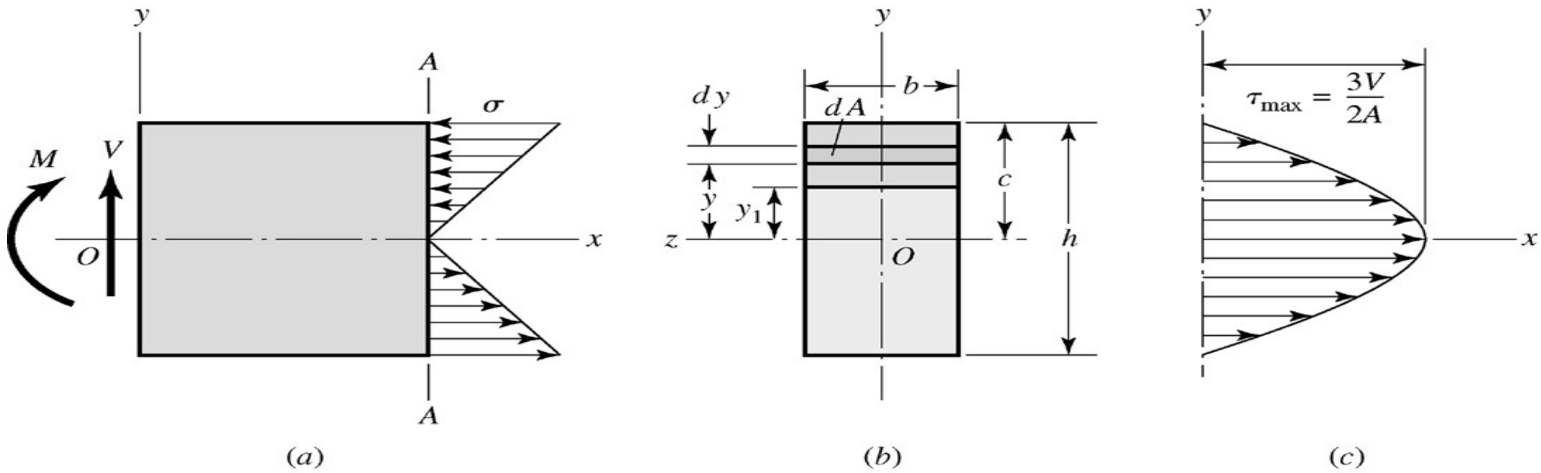
(*b*) Loaded beam showing interfacial sliding and consequent deflection.

(*c*) Shearing stresses develop in glue layers to resist interfacial sliding in laminated beam to reduce beam deflection.

Collins, Fig. 4-5

Stresses

- **Transverse shear stress**



Shigley, Fig. 3-20

$$\tau = \frac{VQ}{Ib} = \frac{V}{Ib}\int_{y_1}^{c} y\,dA$$

V : shear force

Q : area moment about neutral axis

I : moment of inertia

b : width

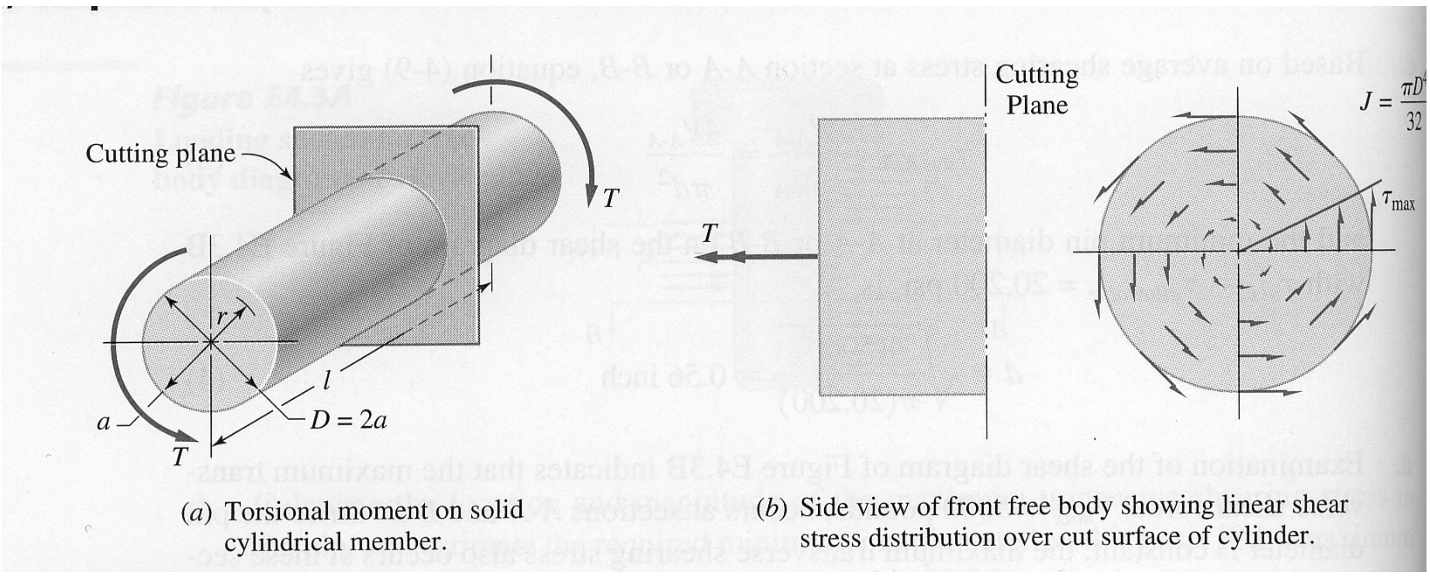
Shigley Table 3-2

Formulas for Maximum Shear Stress Due to Bending

Beam Shape	Formula	Beam Shape	Formula
Rectangular	$\tau_{max} = \frac{3V}{2A}$	Hollow, thin-walled round	$\tau_{max} = \frac{2V}{A}$
Circular	$\tau_{max} = \frac{4V}{3A}$	Structural I beam (thin-walled) (Web)	$\tau_{max} = \frac{V}{A_{web}}$

Stresses

- **Torsion**



Collins, Fig. 4.7

$$\tau = \frac{Tr}{J}$$

T : torque

r : radius

J : polar moment of inertia

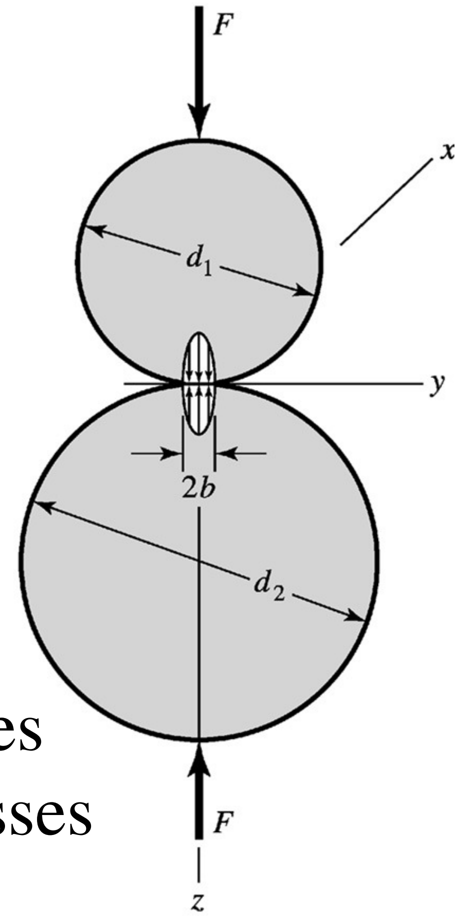
$$J = \frac{\pi D^4}{32}$$

Stresses

- **Contact stresses** (Hertz)

  Important consideration for
  - bearings
  - cams and followers
  - mating gear teeth
  - bodies with curved surfaces

  Small contact area => large stresses



Shigley, Fig. 3-38b)

Sphere: $\sigma_{z\max} = -\dfrac{3F}{2\pi b^2}$

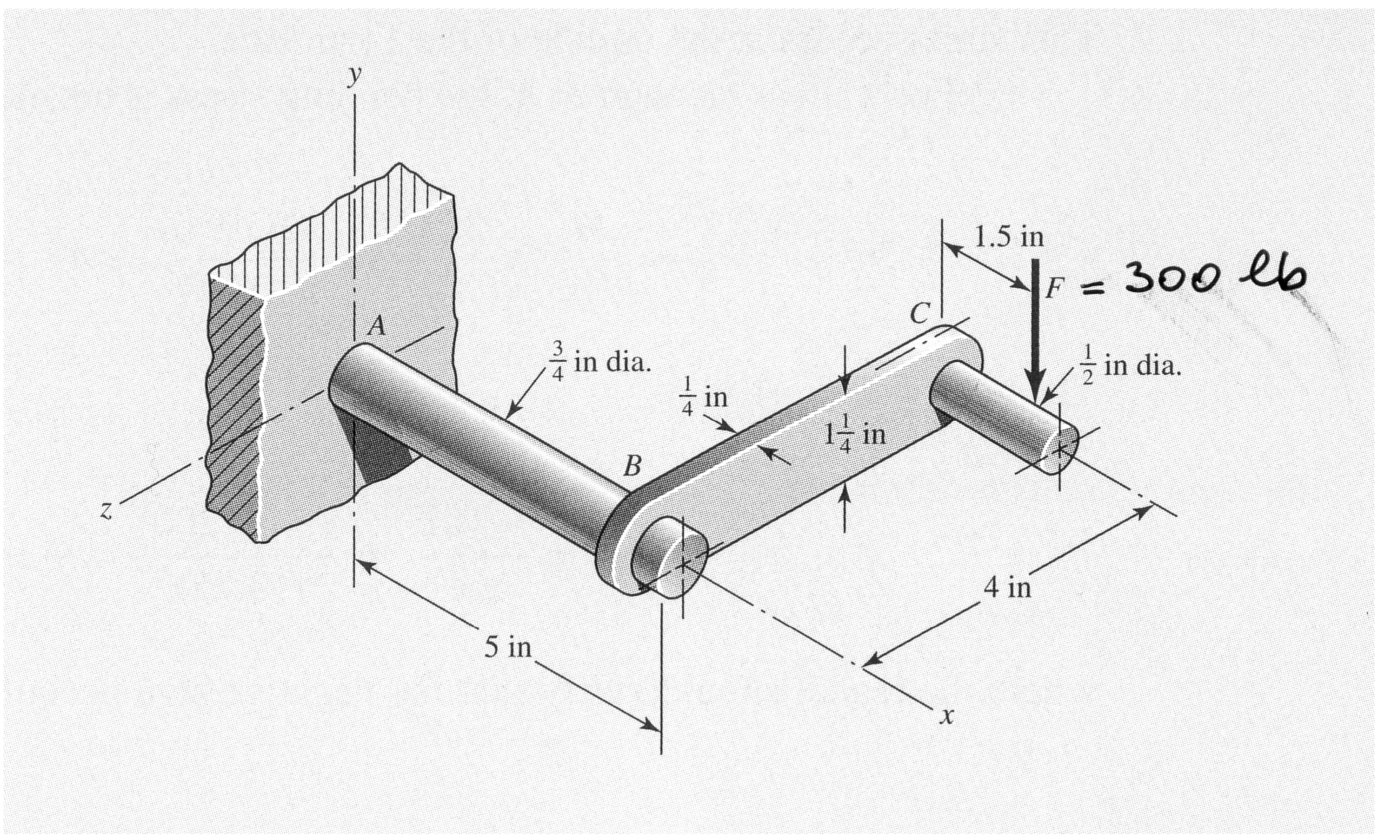
with $b = \sqrt[3]{\dfrac{3F}{8}\dfrac{(1-\nu_1^2)/E_1 + (1-\nu_2^2)/E_2}{1/d_1 + 1/d_2}}$

Cylinder: $\sigma_{z\max} = -\dfrac{2F}{\pi b l}$

with $b = \sqrt{\dfrac{2F}{\pi l}\dfrac{(1-\nu_1^2)/E_1 + (1-\nu_2^2)/E_2}{1/d_1 + 1/d_2}}$

Stresses

- **Combined stresses**
  - Most systems are under the influence of more than one force
  - Determine worst case scenario
  - Find maximum equivalent stress state (Von Mises stress)



Shigley, Fig. 3-22

Combined stresses

- **Equivalent Von Mises stress**

If you already know the principal stresses $\sigma_1,\ \sigma_2,\ \sigma_3$:

$$\sigma' = \sqrt{\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2} \quad \text{in 2D}$$

$$\sigma' = \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}} \quad \text{in 3D}$$

If you only know the applied stresses $\sigma_i,\ \tau_{ij}\ (i, j = x, y, z)$:

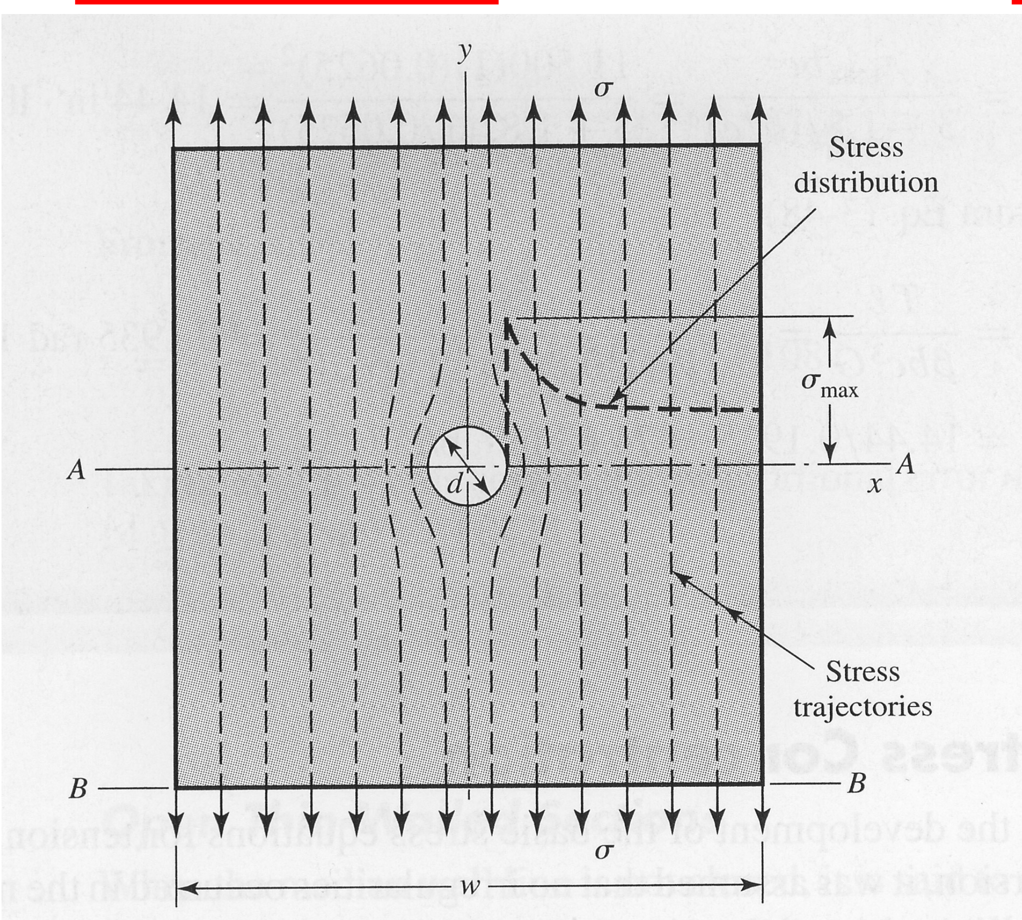
$$\sigma' = \sqrt{\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau_{xy}^2} \quad \text{in 2D}$$

$$\sigma' = \sqrt{\frac{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)}{2}} \quad \text{in 3D}$$

Stress concentrations

- **Geometric stress concentrations**
  - Due to discontinuities of geometry
  - Theoretical values, based on geometry only

$$\boxed{K_t = \frac{\sigma_{max}}{\sigma_{nom}}} \qquad \sigma_{nom} = \frac{F}{(w-d)t} \qquad \boxed{K_{ts} = \frac{\tau_{max}}{\tau_{nom}}}$$



See Tables A-15, A-16

More cases exist

Shigley

Table A–15

Charts of Theoretical Stress-Concentration Factors K_t^* *(Continued)*

Figure A–15–7

Round shaft with shoulder fillet in tension. $\sigma_0 = F/A$, where $A = \pi d^2/4$.

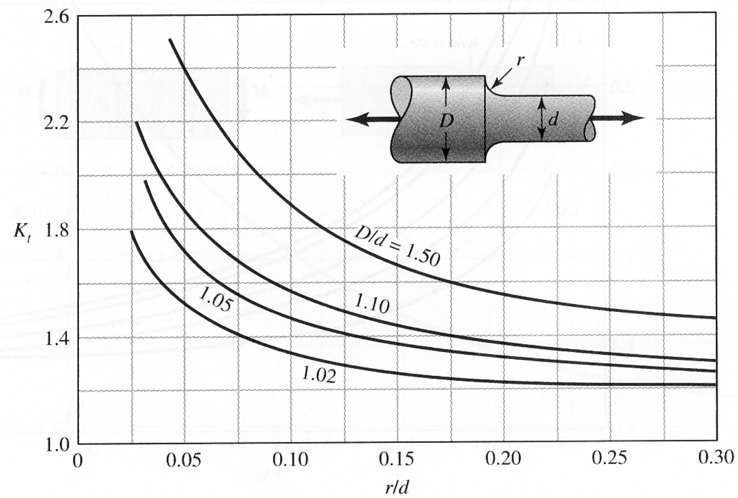


Figure A–15–8

Round shaft with shoulder fillet in torsion. $\tau_0 = Tc/J$, where $c = d/2$ and $J = \pi d^4/32$.

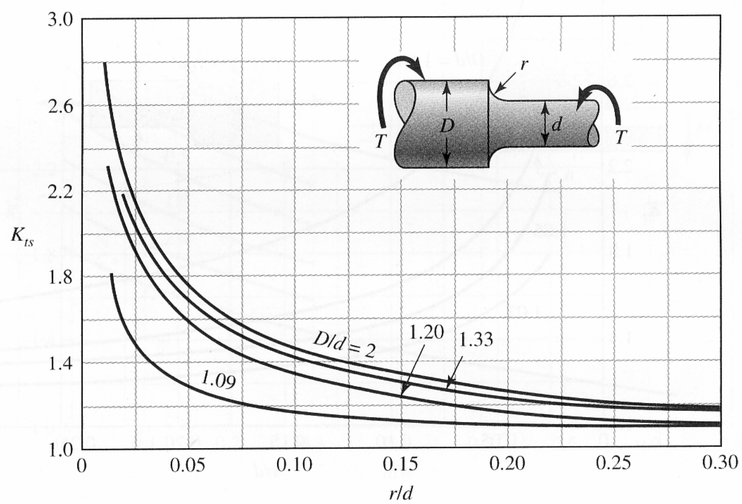
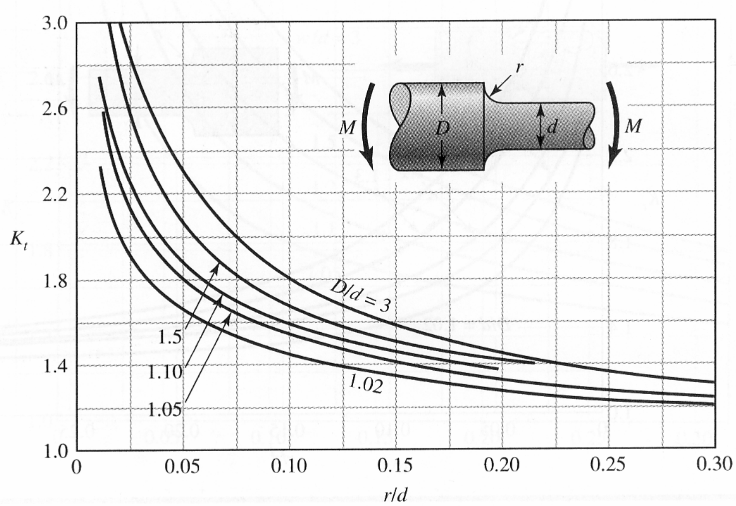


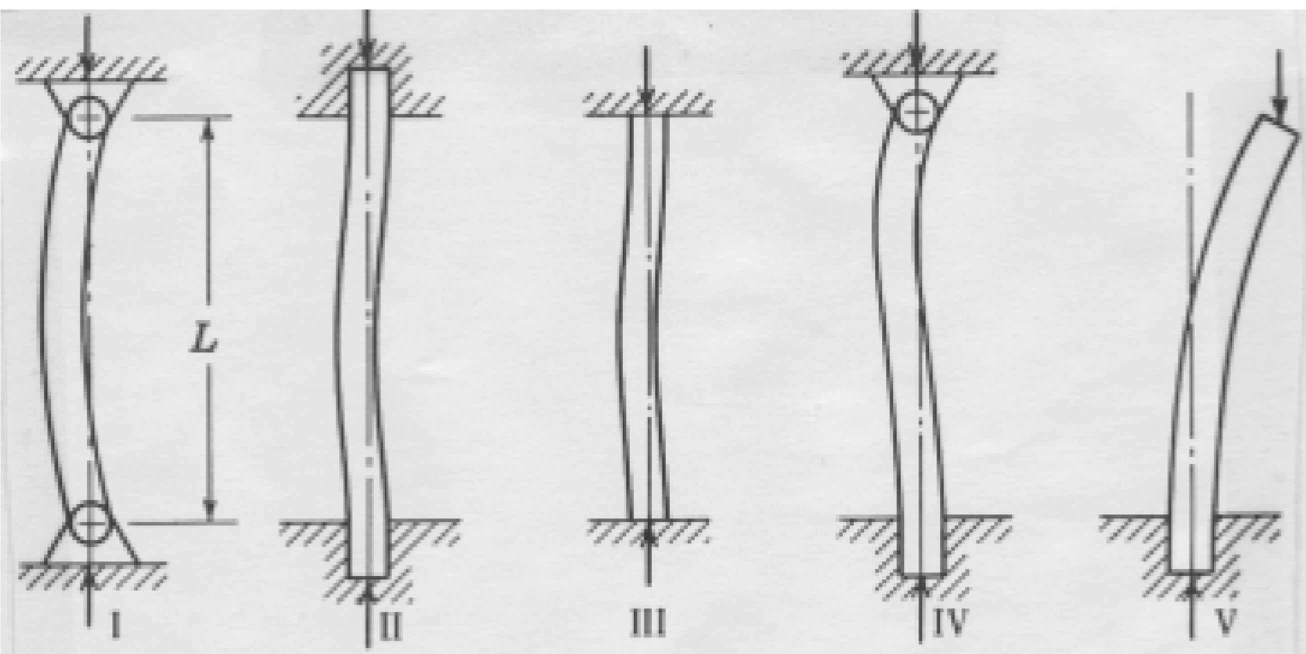
Figure A–15–9

Round shaft with shoulder fillet in bending. $\sigma_0 = Mc/I$, where $c = d/2$ and $I = \pi d^4/64$.



Deflection and stiffness/rigidity

- **Buckling**
  - Columns can collapse in compression
  - Depends on end-condition constant *C*
  - Depends on column geometry



CASE:

I Round ends or pin ends, free but guided
II Fixed ends
III Flat ends
IV One end fixed, other free but guided
V One end fixed, other free

C (See Table 4-2)
1
4
1 to 4
2
0.25

Deflection and stiffness/rigidity

- **Buckling**

Determine $B = \dfrac{S_y l^2}{C\pi^2 E}$ and $k = \sqrt{\dfrac{I}{A}}$

S_y : yield strength of column

k : radius of gyration of column

I : moment of inertia of column

A : area of column cross-section

If $\dfrac{B}{k^2} > 2$ then $F_{cr} = \dfrac{C\pi^2 EI}{l^2}$ (Euler eq.)

If $\dfrac{B}{k^2} \le 2$ then $F_{cr} = AS_y\left(1 - \dfrac{S_y l^2}{4C\pi^2 Ek^2}\right)$ (Johnson eq.)

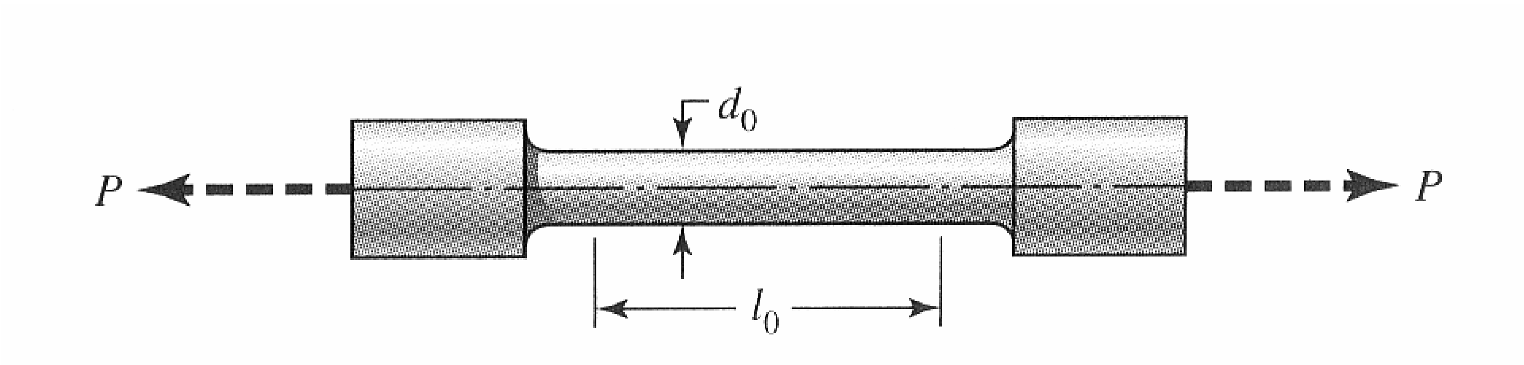
Design considerations: Materials

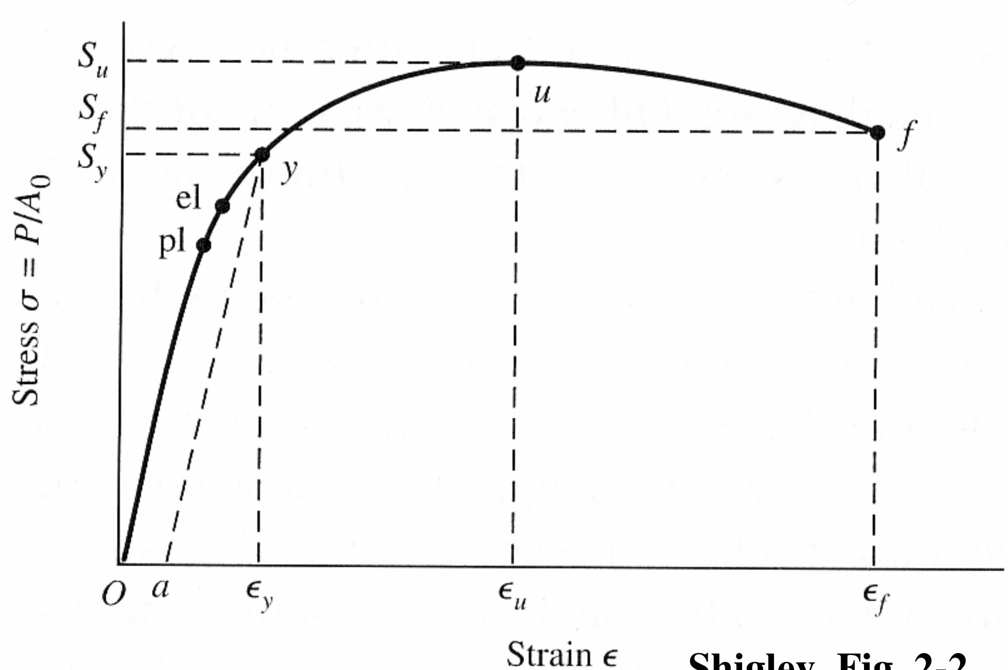
- **Material aspects to consider during the design process**
  - Availability and cost
  - Strength: necessary to prevent failure
  - Rigidity:
    - Deflection limited by service requirements (clearance, alignment)
    - Depends on elastic modulus (not affected by steel strength)
  - Resistance to fatigue:
    - Endurance limit
    - Notch sensitivity
    - Heat treatment required?
  - Resilience
  - Hardness and ductility
  - Weight
  - Electric and thermal properties
  - Resistance to wear and corrosion
  - Manufacturability and machinability
  - Friction (bearings↓, brakes and clutches↑)

Possible materials

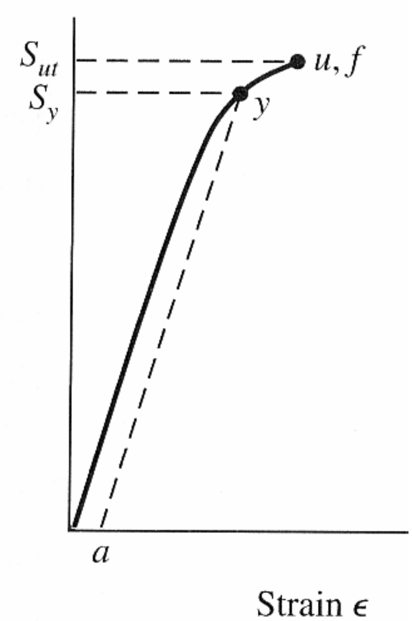
Material	**Applications/properties**
Steel	Crankshaft, piston rods, gears Variable properties alloys
Brass	Moderate strength and ductility Corrosion resistant Good wear qualities (bearing)
Aluminum	High ductility Oxidation resistant Low weight
Titanium	High strength Corrosion resistant Operating temp: -240 to 538 ^{o}C
Plastics	Low weight Manufacturing ease Versatility Corrosion resistant Good shock and vibration resistance Low friction and wear
Composites	Composed of fibres and resins Low weight High strength Plane wings, sports equipment, shafts, hockey sticks

Mechanical properties





Shigley, Fig. 2-2

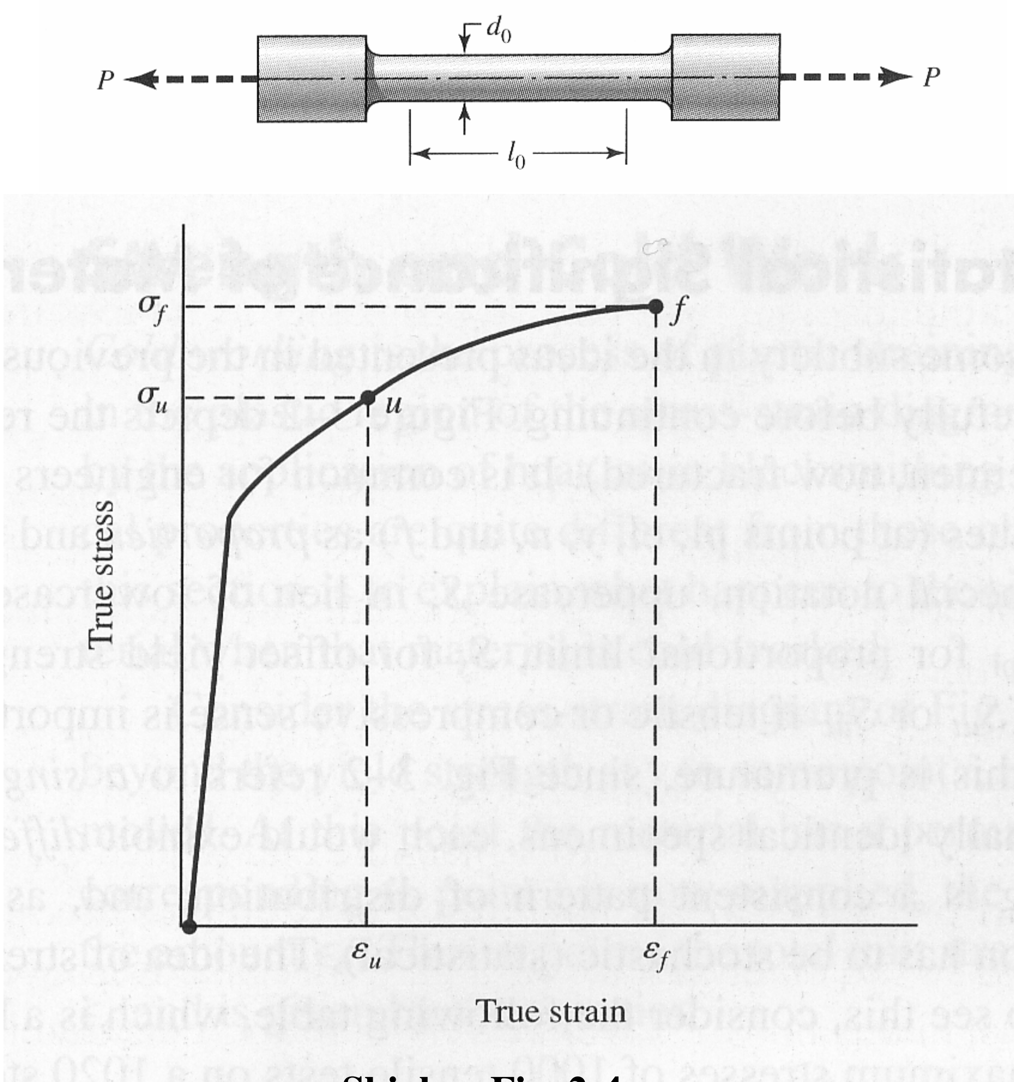


Ductile material

Brittle material

$$\text{Engineering stress: } \sigma = \frac{P}{A_0} \text{ ; engineering strain: } \varepsilon = \frac{l - l_0}{l_0}$$

Mechanical properties



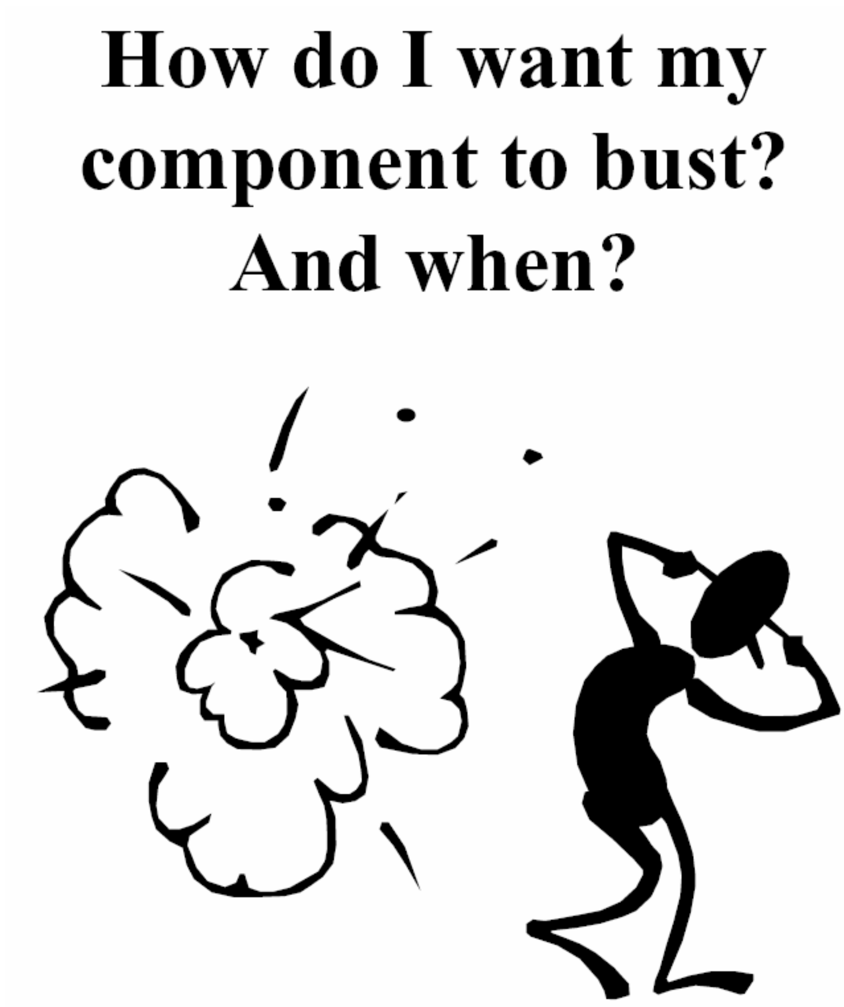
Shigley, Fig. 2-4

Ductile material

$$\text{True stress: } \sigma = \frac{P}{A} \text{ ; true strain: } \varepsilon = \ln \frac{l}{l_0}$$

(Description used for cold work of metals)

Design criteria



- **Failure**: when a component can no longer perform its task

- **Failure modes**
  - **Yielding**: applied stress > yield strength
    => plastic deformation
  - **Fracture** due to
    - Static loading
    - Fatigue loading
    - Impact loading
  - **Excessive elastic deformation**
  - **Wear**: excessive use
  - **Buckling**
  - **Corrosion fatigue and caustic embrittlement** (acid)

- **How to prevent failure?**

  Use a proper safety factor

$$\sigma_{\text{allowable}} = \frac{\sigma_{\text{critical}}}{n}$$

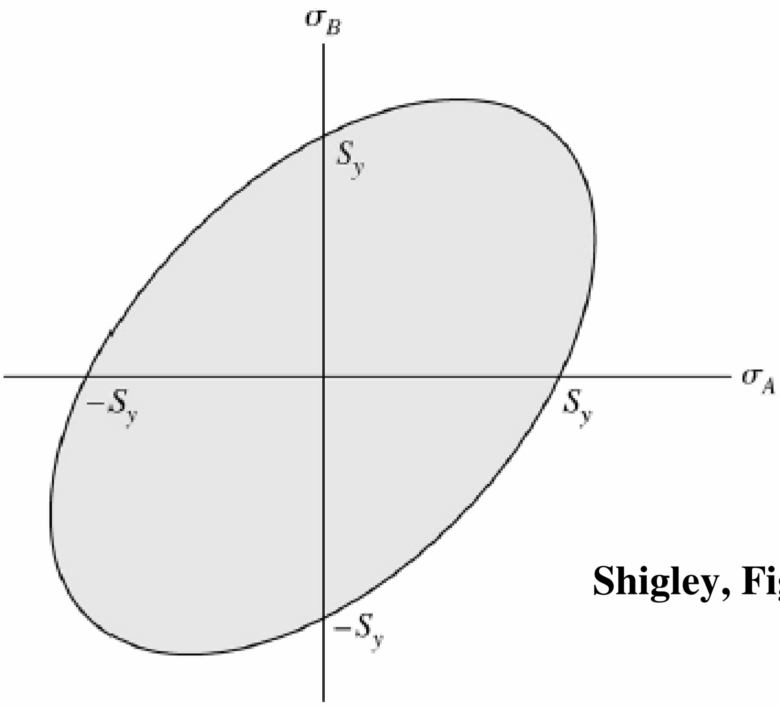
n : factor of safety

σ_{critical} : failure stress, depends on design criteria

$\sigma_{\text{allowable}}$: max stress to be applied (to be safe)

Static loading

- **Distortion-Energy theory**
  (ductile materials)



Shigley, Fig. 5-9

Also called:
- Von Mises theory
- Shear-energy theory
- Octohedral-shear stress theory

Yielding occurs when total strain energy in a volume reaches or exceeds the strain energy corresponding to the tensile or compressive yield strength

Yielding occurs when: $\sigma' \geq S_y$

Von Mises stress :

$$\sigma' = \frac{1}{\sqrt{2}}\begin{bmatrix}(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 \\ +6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)\end{bmatrix}^{1/2}$$

$$S_{sy} = 0.577 S_y$$